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ON AN ARBITRARY FUNCTION IN THE ITERATIVE ANALYTIC SOLUTION OF A GENERAL SYSTEM OF DIFFERENTIAL EQUATIONS

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INTRODUCTION

The classical Picard's existence theorem [4] in the solution of a system of ordinary differential equations has been extended by the introduction of an arbitrary function. It plays a dominant role in the generation of an iterative methodology.

Two diverse points of view are employed for the justification of the extended theorem. The first orientation begins to diverge from the classical one at the point of recognition of the implication of an arbitrary function in the solution. The second point of view [2] has its origin in the concept of a function space in whose postulated structure inheres a unique invariant of a contraction transformation of the space into itself. This invariant is the solution of the system of equations.

Both points of view contribute to an intuitive grasp of the mode of proof of the extended existence theorem and the possible methodologies that flow from these orientations. This is specifically so relative to the more abstract considerations. One of the methodologies briefly discussed in the paper has been used in reference [5] based on a specialized existence theorem.

The paper consists of three parts. The first two deal with the proof of the extended existence theorem. The first part uses classical notions in the generation of the theorem with the necessary deviations to incorporate the concept of an arbitrary function in the sequence of iterative solutions. The base of the development in the second part is the concept of a complete metric space in which a contraction mapping of the space into itself leads to an invariant which on interpretation is the unique solution of the system of differential equations. To gain further intuitive insight into the theorem with its intrinsic methodology, a geometric correspondence to the analysis is observed.

The last part is concerned briefly with two of the many possible methodologies involving the arbitrary function. This part also contains the derivation of an error function which measures the deviation of any element in the sequence of iterative solutions relative to the actual solution of the system of differential equations.

I. THE EXISTENCE OF SEQUENTIAL SOLUTIONS GENERATED BY ARBITRARY FUNCTIONS

The initial phase of the paper will be to extend Picard's existence theorem [4] in the iterative solution of a system of differential equations through the introduction of arbitrary functions whose significance in methodological development will be unfolded. Two distinct processes in the proof of the extended existence will be given. This part of the paper will involve the classical point of view [1] with the necessary deviations relating to arbitrary functions. The second process will involve a more potent abstract base given in the second part.

<u>The Hypothesis on the System of Differential Equations</u>: Let

(1.1) $$\frac{dy^i}{dx} = f^i(x, y^1, y^2, \cdots, y^m), \quad i = 1, 2, 3, \cdots, m$$

be a system of differential equations with initial conditions

(1.2) $$\left[y^i(x)\right]_{x=x_0} = y^i_{(0)}, \quad i = 1, 2, \cdots, m$$

The function f^i, for any i in some space S^{m+1}, is single valued and continuous in a domain $D \subset S^{m+1}$ defined by the inequalities

(1.3) $$|x - x_0| \leq a, \; |y - y^i_{(0)}| \leq b^i; \; x_0, y^1_{(0)}, \cdots, y^m_{(0)} \in D$$

Finally for any two points $(x, Y^1, \cdots, Y^m), (x, y^1, \cdots, y^m) \in D$ the function f^i for any i, satisfies the Lipschitz condition

(1.4) $$\left|f^i(x, Y^1, \cdots Y^m) - f^i(x, y^1, \cdots, y^m)\right| < K^i |Y^i - y^i|$$ for each i

The continuity of $f(x, y^1, \ldots, y^m)$ implies $f^i(x, y_1, \ldots y_m) < M$ on D where M is the greatest of the upper bounds of $f^1, \ldots, f^m$ on D. For the purpose of the discussion, impose a more restrictive condition on x namely that the domain $D' \subseteq D$ be defined by the inequalities,

(1.3') $|x - x_0| \leq h, \; |y^i - y^i_{(0)}| \leq b^i; \; (x_0, y^1_{(0)}, \ldots y^m_{(0)}) \in D'$

where h is the least quantity in the collection $(a, \frac{b^1}{M}, \ldots, \frac{b^m}{M})$

The set of functions $\{y^i(x); i = 1, \ldots, m\}$ necessarily satisfy the system of integral equations

(1.5) $y^i(x) = y^i_{(0)} + \int_{x_0}^{x} f^i[v, y^1(v), \ldots, y^m(v)]\, dv \;, \; i = 1, 2, \ldots, m$

if they also satisfy the conditions (1.1) and (1.2). The well known iterative process in the determination of the unknown function y^i, for any i, implies the generation of the sequence of functions,

(1.6) $\{y^i_n \;;\; i = 1, \ldots, m \;;\; n = 0, 1, 2, \ldots\}$

by means of an extension of the system (I.5), namely

(1.7) $y^i_n(x) = y^i_{(0)} + \int_{x_0}^{x} f^i[v, y^1_{n-1}(v), \ldots, y^m_{n-1}(v)]\, dv \;;\; n = 1, 2, \ldots \;;\; i = 1, 2, \ldots, m.$

Each element y^i_n of the ordered set (1.6) is generated from its immediate predecessor y^i_{n-1} (for any i) by the operational process involved in the $m \times n$ system (1.7). Ultimately $y^i_n(x)$ becomes a function of the initial element $y^i_0(x)$, $(n=0)$, of the sequence (1.6). However the function $y^i_0(x)$ is not defined by (1.7). It follows that each element of the sequence is dependent on any undefined or an initial arbitrary function. For this arbitrary element $y^i_0(x)$ we propose two assumptions:

(1.8) $[y^1_0(x), \ldots, y^m_0(x)]$ is continuous (or even bounded) on $|x - x_0| \leq h$

(1.8') $[y^i_0(x)]_{x = x_0} = y^i_{(0)} \;;\; (y^1_{(0)}, \ldots, y^m_{(0)}) \in D'$

In other words the initial arbitrary functions are assumed bounded and take on the initial values of the function $y^i(x)$ at $x = x_0$ on the domain D'.

Our purpose will be fulfilled if it can be shown that: (1) a set of limiting functions $\{Y^i(x),\ i=1,2,---,m\}$ exist for the ordered aggregate (1.6), (2) the set of limit functions are independent of the collection of initial arbitrary functions $\{y_0^i(x)\}$ (3) $Y^i(x) = y^i(x)$ satisfies the given system of differential equations for an i, (4) $Y^i(x)$ satisfies the initial conditions and (5) are unique relative to the initial values and for any choice of the continuous arbitrary functions. These statements, in fact, constitute, in the rough, the extended existence theorem.

Proof of the Inequalities: The basis in the proof of an extended existence theorem for the limiting function $Y^i(x)$ is to show that the following inequalities

(2.1) $\quad |y_n^i(x) - y_{(0)}^i| \leq b^i;\ i=1,2,---,m,\ n=1,2,---;\ |x-x_0| \leq h$

(2.2) $\quad |y_n^i(x) - y_{n-1}^i(x)| < \dfrac{M\sum_{1}^{m}(|K|)^{n-1}}{n!}|x-x_0|;\ n \geq 1$

are valid. The proof will be given by mathematical induction.

On the assumption (1.8) that the collection of initial arbitrary functions $y_0^i(x)$ of the sequence (1.6) is continuous on the interval $|x-x_0| \leq h$ it follows from (1.7) that for $n=1$, the set $[y_1^1(x),---y_1^m(x)]$ is likewise continous on that interval. The necessary boundedness of the above set is incorporated in the restriction that it is to satisfy the inequality

(2.3) $\quad |y_1^i(x) - y_0^i(x)| \leq b^i,\ |x-x_0| \leq h,\ i=1,2,---,m.$

In view of (1.8'), it follows that

(2.4) $\quad |y_1^i(x) - y_{(0)}^i| \leq b^i,\ |x-x_0| \leq h$

Let us now assume that the expression (2.1) is valid for $(n-1)$ namely that

$$|y_{n-1}^i(x) - y_{(0)}^i| \leq b^i,\quad |x-x_0| \leq h$$

If to this the fact that

$|f^i(x,y^1,---,y^m)| \leq M,\ D' \subset D$ for any i , is added, it then follows that

$$|f^i(v, y'_{n-1}, ---, y^m_{n-1}| \leq M$$

under the s ame conditions. With statement (1.7) in mind it further follows that

$$(2.5)\ |y^i_n(x) - y^i_{(0)}| \leq \int_{x_0}^{x} |f^i[v, y'_{n-1}(v), ---, y^m_{n-1}(v)]\, dv \leq M|x-x_0| \leq Mh \leq b^i \quad \text{for any } i$$

This is so since h (as in (1.3')) is given as the least of the quantities $\frac{b'}{M}, ---, \frac{b^m}{M}$. A further implication is that

$$(2.6) \qquad |f^i[x, y'_n(x), ---, y^m_n(x)]| \leq M, \ |x-x_0| \leq h$$

The second inequality (2.2) is, by comparable means, also shown to be valid. For if it is supposed that this expression is true for $(n-1)$, namely

$$(2.7)\ |y^i_{n-1}(x) - y^i_{n-2}(x)| < \frac{M\bar{K}^{n-2}}{(n-1)!}|x-x_0|^{n-1}, \quad \bar{K} = \sum_{j=1}^{m} K^j; \ |x-x_0| \leq h,$$

then the statement (1.7) generates

$$(2.8) \qquad |y^i_n(x) - y^i_{n-1}(x)| \leq \int_{x_0}^{x} |f^i(v, y'_{n-1}, ---, y^m_{n-1}) - f^i(v, y'_{n-2}, ---, y^m_{n-2})|\, dv$$

The Lipschitz condition implies that

$$|y^i_n(x) - y^i_{n-1}(x)| < K \int_{x_0}^{x} |y^i_{n-1}(v) - y^i_{n-2}(v)|\, dv$$

where $K', K^2, ---, K^m \leq K$. The inequality (2.8) thus leads to the inequality

$$|y^i_n(x) - y^i_{n-1}(x)| < \frac{MK^{n-1}}{(n-1)!} \int_{x_0}^{x} |v-x_0|^{n-1}\, dv = \frac{MK^{n-1}}{n!}|x-x_0| \leq \frac{MK^{n-1}}{n!} h^n$$

In view of the expression (2.1), the above inequality is true for $n=1$. The validity of statement (2.2) is thus established.

<u>The Extended Existence Theorem</u>: A few classical facts in the form of theorems plus the extended existence theorem will be formulated in this section.

With the definition [3] of a uniform convergent series in mind (which of course incorporates the condition that h is independent of x we have the

Theorem 1: The series

(3.1) $$Y^i(x) \equiv y^i_{(0)} + \sum_{r=1}^{\infty}[y^i_r(x) - y^i_{r-1}(x)] \text{ , } i = 1, 2, ---, m$$

is absolutely and uniformly convergent on the interval $|x-x_0| \leq h$

Moreover since $y_1(x)$ is continuous, for each i, implied by the postulated continuity of the initial function $y^i_0(x)$ over the same x interval, it is readily shown by induction that for any n, each element in the sequence of functions (1.6) is continuous. Hence

Theorem 2: The function $Y^i(x)$, given by expression (3.1) for any i, is continuous on the interval $|x-x_0| \leq h$. In fact the theorem is also manifest from the character of the series representing it over the interval.

Consider the expansion of $y^i_n(x)$ by the identity
$$y^i_n(x) - y^i_0(x) \equiv y^i_1(x) - y^i_0(x) + y^i_2(x) - y^i_1(x) + --- + y^i_n(x) - y^i_{n-1}(x), \; i = 1, 2, ---, m.$$
This leads to the finite sum,

(3.2) $$y^i_n(x) = y^i_0(x) + \sum_{r=1}^{n}[y^i_r(x) - y^i_{r-1}(x)]$$

A comparison of (3.1) and (3.2) as $n \to \infty$ generates the limit function

(3.3) $$Y(x) = \lim_{n \to \infty} y_n(x)$$

This allows the formulation of

Theorem 3: A limit function $Y^i(x)$ exists, for each i, of the ordered set of iterative functions $\{y^i_n(x); i = 1, 2, ---, m; n = 0, 1, 2, ---\}$. Every element of the set is continuous on the interval $|x-x_0| \leq h$.

The following theorem is readily shown.

Theorem 4: The limit function $Y^i(x)$ is a solution of the integral equation,
$$Y^i(x) = y^i_{(0)} + \int_{x_0}^{x} f^i[v, Y^1(v), ---, Y^m(v)]\,dv, \; i = 1, 2, ---, m.$$
By means of definition (3.1) and the Theorem 3,
$$Y^i(x) = \lim_{n \to \infty} y^i_n(x) = y^i_0 + \lim \int_{x_0}^{x} f^i[v, y^1_{n-1}(v), ---, y^m_{n-1}(v)]\,dv$$

To show that

$$\lim_{n\to\infty}\int_{x_0}^{x} f^i(v, y'_{n-1}, ---, y^m_{n-1})\,dv = \int_{x_0}^{x}\lim_{n\to\infty} f^i(v, y'_{n-1}, ---, y^m_{n-1})$$

use is made of the Lipschitz condition in the form

$$\left|\int_{x_0}^{x}\left[f^i(v, Y', ---, Y^m) - f^i(v, y'_{n-1}, ---, y^m_{n-1})\right]dv\right| < K\int_{x_0}^{x}\sum_{j=1}^{m}\left|Y^j(v) - y^j_{n-1}(v)\right|dv$$

where $K', ---, K^m \leq K$ But

$$\sum_{j=1}^{m}\int_{x_0}^{x}\left|Y^j(v) - y^j_{n-1}(v)\right|dv < \epsilon_n|x - x_0| \leq \epsilon_n h$$

where ϵ_n is independent of x and $\lim_{n\to\infty}\epsilon_n = 0$. It follows that

$$Y^i(x) = \lim_{n\to\infty} y^i_n(x) = y^i_{(0)} + \int_{x_0}^{x}\lim_{n\to\infty} f^i(v, y'_{n-1}, ---, y^m_{n-1})\,dv \tag{3.4}$$
$$= y^i_{(0)} + \int_{x_0}^{x} f^i(v, Y', ---, Y^m)\,dv\,, \quad i = 1, 2, ---, m$$

We now prove

Theorem 5: The solution $Y^i(x)$ of the integral equation (3.4) is the same as the solution $y^i(x)$ of the system of differential equations (1.1) and satisfies the same boundary conditions, namely

$$Y^i(x) = y^i(x);\ |x - x_0| \leq h;\ \left[Y^i(x)\right]_{x=x_0} = \left[y^i(x)\right]_{x=x_0} = y^i_{(0)}\ ;\ i = 1, 2, ---, m \tag{3.5}$$

Since $f^i[v, Y'(v), ---, Y^m(v)]$, as given in (3.4), is continuous on $R' \subset R$, it follows that

$$\frac{dY^i(x)}{dx} = \frac{d}{dx}\int_{x_0}^{x} f^i(v, Y', ---, Y^m)\,dv = f^i(x, Y', ---, Y^m)\,, \quad i = 1, \cdots, m.$$

So that $Y^i(x)$ and $y^i(x)$ satisfy the same system of differential equations

$$\frac{dz^i}{dx} = f^i[x, z'(x), ---, z^m(x)] \tag{3.6}$$

To show that the limit function fulfills the same initial conditions as $y^i(x)$, substitute the value $x = x_0$ in (3.4). This leads to the initial stipulations

$$\left[Y^i(x)\right]_{x=x_0} \equiv Y^i(x_0) = y^i_{(0)}\ , \quad i = 1, 2, ---, m$$

The uniqueness of the limit function is given by means of two theorems that follow. The first theorem specifies uniqueness relative to initial conditions and the second asserts the independence of the limit function with regard to the initial function of the sequence of iterative functions. The latter theorem, as will be shown in the sequel, is basic to an approximation methodology.

Theorem 6: The limit function $Y^i(x) = \lim_{n\to\infty} y^i_n$, $i=1,2,---,m$ is a unique solution of the given system of differential equations with the associated specified initial conditions. The validity of this theorem follows immediately from the observation that the specification of the Lipschitz condition implies the theorem.

Theorem 7: The limit function $Y^i(x)$ is independent of the choice of the initial function $y_0(x)$ in the iterative sequence of functions $\{y^i_n(x);\, i=1,2,---,m;\, n=0,1,2,---\}$.

Suppose that the two functions $Y^i(x)$ and $\bar{Y}^i(x)$ are the limits of the respective sequences $(y^i_0, y^i_1, y^i_2, ---)$ and $(\bar{y}^i_0, \bar{y}^i_1, \bar{y}^i_2, ---)$. Each sequence is generated by the respective initial functions y^i_0, $\bar{y}^i_0$. By Theorem 4, each of the limit functions satisfies the integral equation

$$Y^i(x) = y^i_{(0)} + \int_{x_0}^{x} f^i[v, Y^1(v), ---, Y^m(v)]\,dv;\quad \bar{Y}^i(v) = y^i_0 + \int_{x_0}^{x} f^i[v, \bar{Y}^1(v), ---, \bar{Y}^m(v)]\,dv$$

respectively, where it is assumed that the s ame initial conditions, $[Y^i(x)]_{x=x_0} = y^i_{(0)}$ subsist. Since the form of the integral equations are the same, it follows from Theorem 5 that each limit function satisfies the same system of differential equations (3.6) with the same boundary conditions $[z^i(x)]_{x=x_0} = z^i_{(0)}$. Finally by Theorem 6, this system with the associated boundary conditions implies a unique solution. Hence $Y^i(x) = \bar{Y}^i(x)$; $i=1,2,---,m$; $|x-x_0| \le h$.

The preceding theorems will now be assembled into a single one with the hypothetical specifications incorporated. This summarizing statement is an extension of the classical existence theorem (Picard) for the solution of a system of differential equations with its associated boundary conditions. The extension specifies the solution of the system as a limit function of a sequence of functions each of which (except the initial one) is a solution of a system of integral equations. The initial element of the sequence is arbitrary whereas the remaining functions of the ordered set are ultimately dependent on it. However, the limit function, itself, is independent of any specific choice in the replacement of this initial arbitrary function.

Extended Theorem 8: (A) The continuous limit function $Y^i(x)$, $i=1,2,\dots,m$; $|x-x_0|\le h$ of the sequence of continuous functions $\{y^i_n(x)$; $i=1,2,\dots,m$; $n\ge 0$; $|x-x_0|\le h$ is a unique solution (for each i) of the system of differential equations $\frac{dy^i}{dx}=f^i(x,y^1,\dots,y^m)$ relative to the initial conditions $[y^i(x)]_{x=x_0}=y^i_{(0)}$ and ( ) is independent of the initial arbitrary function $y^i_0(x)$ in the sequence $\{y^i_n(x)\}$. Each element in the ordered set (except the initial one) satisfies a system of integral equations of the form

$y^i_n(x) = y^i_{(0)} + \int_{x_0}^{x} f^i(v, y^1_{n-1},\dots,y^m_{n-1})\,dv.$ (B) The function $f^i(x,y^1,\dots,y^m)$, for each i, is assumed continuous on $R'\subset R$ where the domains R and R' are defined respectively by the inequalities $|x-x_0|\le h$, $|y^i - y^i_{(0)}|\le b^i$; $(x_0,y^1_{(0)},\dots,y^m_{(0)})\in R'\subset R$. The quantity h is defined as the smallest quantity in the collection $(a, \frac{b^1}{M},\dots,\frac{b^m}{M})$ and where M is the largest in the set of upper bounds of the functions $f^i(x,y^1,\dots,y^m)$.

Two items in the extended theorem are worthy of note. (1) The hypothesis that the functions f^i are continuous over the specified domain is too restrictive. It may be replaced by the conditions that the functions be bounded with no change in the proof. However, the hypothesis of continuity will, for this paper, prove to be sufficiently useful. (2) The three properties of the initial function $y^i_0(x)$ in the iterative sequence $(y^i_0, y^i_1, y^i_2,\dots)$, namely that $y^i_0(x)$ is arbitrary (but assumed continuous on the defined x-interval), that the remaining functions $(n>0)$ are dependent on it and that the limit function $Y^i(x)=\lim_{n\to\infty} y^i_n(x)$, contrawise, is an invariant relative to it, these properties are precisely the ones which will allow the evolution of a general iterative methodology in the solution of systems of differential equations. This subject will be developed briefly in a section of the final part of the paper.

II THE SPATIAL STRUCTURE OF THE EXTENDED THEOREM

An elegant mode in the formulation of the extended existence theorem is to exhibit the more abstract form which the proof evolves. This more general presentation will generate a new point of view and so give larger scope to applications of an iterative process in the solution of systems of differential equations.

<u>Introduction to the Concepts</u>: The concepts involved in the subject matter are the notions of a complete metric space and the contraction mapping of that space into itself. The definitions and some of the consequences are taken from reference [2] .

A metric space consists of two items: a set X of elements (points) and a single valued, non negative real function $\rho(x,y)$; $x,y \in X$ called the metric (distance) of the space. This function satisfies three axioms given in the above reference. If R is a metric space, a sequence $\{x_n\}$ of points $\in R$ is called fundamental if it converges to some limit (i.e., if it satisfies the Cauchy criterion). The metric space R is said to be complete if every fundamental sequence in R converges to an element $\in R$.

A mapping A of an arbitrary metric space into itself is said to be a contraction if there exists an $\alpha < 1$ such that

$$\rho(Ax, Ay) \leq \alpha \rho(x,y) \; ; \; x,y \in R$$

It is readily verifiable that a contraction transformation is a continuous function. The following theorem is basic to a complete metric space [2, p 43]:

Every contraction mapping A defined in a complete metric space R has a unique invariant point. Thus the equation $Ax = x$ has a unique solution, namely the point $x \in R$ under the transformation A if it is transformed into itself, is transformed uniquely.

The Space of the System of Differential Equations: The initial assumptions on the system of differential equations are the same as that given in part I but with a more abstract orientation. Thus consider space with the ordered $(m+1)$- tuple $(x, y^1, ---, y^m$ as a generic element (point) (x,y) of that space. To each point is associated a collection of continuous functions $\{f^i(x, y^1, ---, y^m; i = 1, 2, ---, m\}$ on a region $D \subset R^{m+1}$ which defines the components of the direction element at any point (x,y) in that region, namely

(4.1) $$\frac{dy^i}{dx} = f^i(x, y^1, ---, y^m) \ , i = 1, 2, ---, m$$

with an associated set of initial conditions

(4.2) $$[y^i(x)]_{x=x_0} = y^i_{(0)} \ , i = 1, 2, ---, m \ ; (x_0, y^1_{(0)}, ---, y^m_{(0)}) \in D.$$

The function f^i for any i, is further restricted by a Lipschitz inequality,

(4.3) $$|f^i(x, y_1) - f^i(x, y_2)| \leq max_{x,i} |y_1^i - y_2^i|$$

and where M is defined as the greatest of the upper bounds of $f^1, ---, f^m$ on D.

On the basis of the above assumptions, the validity of the extended theorem 8 will be shown in a different perspective. Again it will be seen that a set of unique continuous limit functions of a sequence of continuous functions (over a specified range) exist which satisfy the system (4.1) and the associated conditions (4.2) and which set is independent of the collection of arbitrary functions, involved in the sequence, as an initial element.

<u>Contraction Mapping in a Function Space</u> [2, p48]: Introduce the metric space C^m with a general element y(x) defined as a continuous function over a specified x-range. The element y(x) is given by an ordered m-tuple of functions

(5.1) $$y(x) = [y^1(x), \cdots, y^m(x)],$$

each component of which is continuous for all x in the interval $|x-x_0| \leq h$ and such that $|y^i(x)| \leq kh$, k a positive quantity, $i = 1, 2, \cdots, m$. The quantity $h > 0$ is so chosen that the two conditions

(5.2) $[x, y(x)] \in D$ if $|x - x_0| \leq$ and $|y^i - y^i_{(0)}| \leq Kh$

(5.3) $Mh < 1$

The distance function ρ, chosen for the metric space C^m, is defined by the statement

(5.4) $\rho[y_p(x), y_s(x)] = \max_{x,i} |y^i_p - y^i_s(x)|$, $i = 1, 2, \cdots, m$; $|x - x_0| \leq h$

where $y_p \equiv (y^1_p, \cdots, y^m_p)$, $y_s \equiv (y^1_s, \cdots y^m_s)$ are any two points $\in C^m$

Finally, it is readily shown [2, p37] that the given metric space is complete. We may summarize by stating that the function space C^m is a complete metric configuration whose elements are continuous functions over a defined x-range.

If a defined mapping A is introduced into the space C^m, it will have to be shown that this mapping function is a contraction if the basic theorem of a contraction transformation of a complete metric space into itself, is to be applied. The procedure, in what follows, is to introduce by definition just such a transformation.

The transformation A on C^m implies the operational equation.

(5.5) $y_r(x) = A y_s(x)$; $y_s(x)$ a given point, $y_r(x)$ its image; $y_s, y_r \in C^m$

The general relation (5.5) is defined to take the specific form of an integral equation, (5.6) $\quad y_n(x) = y_{(0)} + \int_{x_0}^{x} f[v, y_{n-1}(v)]\,dv \quad , n = 1, 2, ---$

This operational equation implies a countable ordered set of functions

(5.6') $\quad [y_0(x), y_1(x), y_2(x), ---]$

Each element $y_n(x)$ in the sequence is generated by its immediate predecessor y_{n-1} (but for the initial function $y_0(x)$) by the defined transformation (5.6).

The initial element $y_0(x)$ is manifestly undefined (arbitrary). However, we will endow it with one property, namely that it be continuous over the interval $|x - x_0| \le h$. This implies, by virtue of statement (5.6), that the remaining functions $y_1(x), y_2(x), ---$ of the sequence (5.6') are likewise continuous for the specified x-interval.

Since each element $y(x) \in C^m$ is defined as an ordered m-tuple, $y_n \in C^m$ is expressed as

$$y_n = (y_n^1, y_n^2, ---, y_n^m) \;,\; y_n \in C^m \;,\; n = 0, 1, 2, --- .$$

It follows that the integral equation (5.6) may be written as a system of componental equations. Thus

(5.7) $\quad y_n^i(x) = y_{(0)}^i + \int_{x_0}^{x} f^i[v, y_{n-1}^1(v), ---, y_{n-1}^m(v)]\,dv;\; i = 1, 2, ---, m;\; n \ge 1;\; |x - x_0| \le h$

We now show that the sequence of integral equations (5.7) represents a contraction mapping of the complete metric space C^m into itself.

The statement (5.7) leads to the expression

(5.8) $\quad y_n^i(x) - y_{n-1}^i(x) = \int_{x_0}^{x} [f^i(v, y_{n-1}^1, ---, y_{n-1}^m) - f^i(v, y_{n-2}^1, ---, y_{n-2}^m)]\,dv$

The application of the Lipschitz condition (4.3) to the right member gives

(5.9) $\quad \int_{x_0}^{x} |f^i(v, y_{n-1}^1, ---, y_{n-1}^m) - f^i(v, y_{n-2}^1, ---, y_{n-2}^m)|\,dv \le \int_{x_0}^{x} M \max_{x,i} |y_{n-1}^i - y_{n-2}^i|\,dv$

$$= M \max_{x,i} |y_{n-1}^i - y_{n-2}^i|\,|x - x_0| \le M h \max_{x,i} |y_{n-1}^i - y_{n-2}^i|$$

The equality (5.8) thus turns into the inequality

(5.10) $\quad |y_n^i(x) - y_{n-1}^i(x)| \le M h \max_{x,i} |y_{n-1}^i(x) - y_{n-2}^i(x)|;\; 1 \le i \le m;\; |x - x_0| \le h$

where by the assumption (5.3) $Mh<1$

Since the difference $(y_n^i(x) - y_{n-1}^i(x))$ of the left member of (5.10) is continuous for any i, n and x in its given interval, it follows that this difference has a numerical maximum for some i and x. So that the expression $\max_{x,i}|y_n^i(x) - y_{n-1}^i(x)|$ would still be $\leq$ right member of (5.10). Thus the statement, derived from (5.10),

$$(5.11)\quad \max_{x,i}|y_n^i(x) - y_{n-1}^i(x)| \leq Mh \max_{x,i}|y_{n-1}^i(x) - y_{n-2}^i(x)|\ ,\ Mh \leq 1$$

is valid.

In view of the definition (5.4) for the metric ρ of C^m, the inequality (5.11) may be written as

$$(5.12)\qquad \rho(y_n^i, y_{n-1}^i) \leq Mh\rho(y_{n-1}^i, y_{n-2}^i)\ ;\ Mh<1$$

But the expression (5.12) is, by definition, the condition that the mapping function A as given by (5.6) or in the componental form (5.7), be a contraction mapping of the space C^m into itself. In fact (5.12) may take on the more general form

$$(5.12')\quad \rho(Ay_{n-1}^i, Ay_{n-2}^i) \leq \alpha\, \rho(y_{n-1}^i, y_{n-2}^i),\quad \alpha = Mh<1\ ;\ y_{n-1}^i, y_{n-2}^i, Ay_{n-1}^i, Ay_{n-2}^i \in C^m$$

The space C^m may now be fully characterized by the

<u>Theorem1</u>: The set C^m of continuous function elements $y(x) = [y^1(x), ---, y^m(x)]$ $|x-x_0| \leq h$ is a complete metric space with an allowable contraction transformation which leaves the space invariant. The metric function is defined by formula (5.4) and the contraction mapping by (5.7) which generates the sequence

$$[y_0^i(x),\ y_1^i(x), ---]\ ;\ i=1,2,---,m\ ;\ |x-x_0| \leq h$$

of continuous functions where the initial collection $[y_0^1(x), ---, y_0^m(x)]$ is arbitrary but with defined continuity over the x-interval.

The Application of the Principle of Contraction Mapping: The system of differential equations (4.1) together with the associated set of boundary conditions (4.2) is equivalent to the system of integral equations

(6.1) $y^i(x) = y^i_{(0)} + \int_{x_0}^{x} f^i[v, y^1(v), ---, y^m(v)]\, dv \; ; \quad i = 1, 2, ---, m$

We must now show the type of relation that exists between the solution of (4.1), (4.2); (6.1) and the solution $y^i_n(x)$ of the contraction mapping given by the expression (5.6) or (5.7).

We repeat the statement of the basic theorem for a contraction mapping A [2, p43]. Every contraction mapping A defined in a complete metric space C^m has one and only one invariant point, namely that the equation $y(x) = A\,y(x),\ y(x) \in C^m$ has a unique solution in C^m. This theorem applied to the contraction operation given by the integral expression (5.7) in its limiting form,

(6.2) $\lim_{n\to\infty} y^i_n(x) = y^i_{(0)} + \int_{x_0}^{x} \lim_{n\to\infty} \left\{ f^i[v, y^1_{n-1}(v), ---, y^m_{n-1}(v)] \right\} dv$

thus necessarily leads to the unique solution

(6.3) $Y^i(x) = \lim_{n\to\infty} y^i_n(x) \; ; \; |x - x_0| \le h \; ; \; i = 1, 2, ---, m$

and which satisfies the initial conditions

$$[Y^i(x)]_{x=x_0} = y^i_{(0)}$$

The symbol $Y^i(x)$ denotes the limit function. Stated otherwise, the sequence of functions

(6.4) $\left\{ y^i_n(x) \; ; \; i = 1, 2, ---, m \; ; \; n = 0, 1, 2 --- \; ; \; |x - x_0| \le h \; ; \; y^i_n(x) = (y^1_n, ---, y^m_n) \in C^m \right.$

with $y^i_0(x)$ arbitrary but continuous, has (by virtue of the fact that C^m is a complete space) a limit and since the mapping function (which generates this sequence) is a contraction, it follows that the limit function $Y^i(x)$ is unique and so independent of the character of the initial element $y^i_0(x)$ for any i.

The relation between the unique limit function $Y^i(x) = \lim_{n\to\infty} y^i_n(x)$ of the sequence (6.4) and the unique solution $y(x)$ of the system of differential equations (4.1) and its associated boundary conditions (4.2), is now readily established. If the integral expression (6.1) is viewed as a contraction mapping, namely $y^i(x) = A\, y^i(x)$ then the unique limit function $Y^i(x)$ of the sequence (6.4) must be the same as the $y^i(x)$ of (6.1), since any arbitrary sequence in a complete metric space with a contraction transformation leads to a unique limit. But the solution $y^i(x)$, for any i, of (6.1), now viewed as a system of integral equations, has the same unique solution as the solution $y^i(x)$ of the system (4.1), (4.2). It follows that

$$(6.5)\quad Y^i(x) = y^i(x),\ i=1,2,\cdots,m;\ |x-x_0|\le h;\ \left[Y^i(x)\right]_{x=x_0} = \left[y^i(x)\right]_{x=x_0} = y^i_{(0)}$$

A final observation is made to emphasize (a fact noted more than once) the significance of the initial arbitrary function $y_0(x)$ of the sequence (6.4) and which thereby also allows us to make the necessary connection with the extended existence theorem of part I. The elements in the sequence (6.4) of approximate solutions, generated in order (but for the initial element) by the contraction transformation (or the system of integral equations (5.7)), are, as has been noted, dependent on this initial function. Its arbitrary character (modified by the demand that it be continuous) implies (by the properties of the C^m space) that for each choice of the initial function $y^i_0(x)$ a new sequence is evolved and that there exists as many sequences as there are choices. However, we are told that the limit function for any sequence is the same (unique), namely that $Y^i(x)$ is invariant for every sequence generated by the contraction mapping and the initial function. This observation permits us to write verbatum the extended existence theorem of part I and to assert its validity.

A Geometric Interpretation of Contraction in C : A spatial representation of the analysis involved in the formulation of the extended theorem will be given in what follows. The purpose is to point towards an isomorphism whose existence allows an intuitive insight into the analysis and leads to more extensive analogies in application.

The general element $y(x) = [y^1(x), ---, y^m(x)]$ of the metric space C^m is a continuous curve C given by the m parametric equations

(7.1) $y^i = y^i(x)$, $i = 1, 2, ---, m$

where $[y^i(x)]_{x=x_0} = y^i_{(0)}$, $i = 1, ---, m$ is an initial point on C. The parameter x satisfies the inequality $x_0 - h \leq x \leq x_0 + h$ and is the range for which the curve C is continuous. The metric function ρ of C^m, given by (5.4), is the distance between any two points, say y_p and y_s , on the curves C_p [$y^i_p = y^i_p(x)$ and C_s [$y^i_s = y^i_s(x)$ respectively and whose coordinates satisfy the definition (5.4) for ρ .

From the fact that C^m is a metric space, it necessarily follows that the space is also complete, namely that if a sequence of curves $(C_0, C_1, C_2, ---)$ converges to some limiting curve C (a fundamental sequence) then C $\in$ C^m. A metric space is also a complete space.

If on this complete metric space an independent transforming constraint is imposed, namely a contraction mapping A of the space into itself, the structure of the space becomes so concretely organized that for any arbitrary fundamental sequence of curves in the space, one and only one limiting curve exists belonging to that space. This limit curve is the unique invariant of the contraction mapping function of the complete metric space C^m transformed into itself.

If the transformation A is specifically defined by the system of integral equations (5.6) or (5.7) (and this definition has been justified) then for any arbitrary curve C_0 , given by $y_0^i = y_0^i(x)$, $i = 1, 2, ---, m$ a convergent (fundamental) sequence $(C_0, C_1, C_2, ---)$ of curves of C^m

is evolved (generated by (5.6) or (5.7)) whose limiting curve C is the unique invariant of the contraction mapping A of C^m. Thus to any arbitrary curve C_0 will correspond a family of sequential curves C_i, $i=1,2,---,m$ (generated by the mapping A) whose limiting curve C is the unique invariant of the transformation. The sequence of curves (including the initial one) and the limit curve are all elements of C^m.

If it is further postulated that the initial arbitrary curve passes through a given initial point $[y^i(x)]_{x=x_0}=y^i_{(0)}$, $i=1,---,m$ and that $C_0\,[y_0=y^i_0(x)]$ is continuous over the interval $|x-x_0|\leq h$ then the sequential family becomes a pencil of continuous curves which converge to the unique invariant continuous curve C which likewise passes through the given initial point. To each choice of the arbitrary curve will correspond a sequential pencil of curves and there will exist as many pencils as there are arbitrary continuous curves on an interval. But whatever the choice of the arbitrary function (and so of a corresponding pencil), the limit curve will be invariant. The limit curve will be a function only of the contraction mapping. The limit function is thus a part of the structure of the space C^m since it admits into its structure this type of transformation.

The final step is to associate the invariant curve C of C and given by the parametric equations

$$(7.2)\qquad Y^i = Y^i(x) = \lim_{n\to\infty} y^i_n(x) \quad , i=1,---,m$$

with a curve, given by

$$(7.3)\quad \begin{cases} y^i = y^i(x) \; ; [y^i(x)]_{x=x_0}=y^i_{(0)} \; ; i=1,---,m \\ x = x \end{cases}$$

whose line element is spcified by the direction

$$(7.4)\qquad \frac{dy^i}{dx} = f^i(x, y^1, ---, y^m) , \; [y^i(x)]_{x=x_0}=y^i_{(0)}$$

at a generic point (x,y), $y = (y^1, \ldots, y^m)$ of the space R^{m+1}. It is supposed that the function f^i fulfills all the conditions of the initial hypothesis.

The equation (7.4) has for its unique solution the curve in R^{m+1} given by the equations (7.3). This solution is correpondingly expressed implicitly by the system of integral equations (6.1). This implicit statement is given explicit formulation by the converging sequence of approximate solutions $[y_0^i(x), y_1^i(x), \ldots]$ of the system of integral equations (5.7), namely

$$y^i(x) = \lim_{n \to \infty} y_n^i(x), \quad n = 0, 1, 2, \ldots \tag{7.5}$$

If now the same system (5.7) is viewed as a contraction mapping of the complete metric space C^m into itself, then the invariant curve

$$Y^i = Y^i(x), \quad [Y^i(x)]_{x=x_0} \equiv [y^i(x)]_{x=x_0} = y^i_{(0)}$$

of the transformation is also the limit of the pencil of curves $(C_0, C_1, \ldots)$ given by the same fundamental sequence $[y_0(x), y_1(x), \ldots]$. So that

$$Y^i(x) = \lim_{n \to \infty} y_n^i(x) = y^i(x); \quad [y^i(x)]_{x=x_0} = y^i_{(0)} \tag{7.6}$$

where the initial arbitrary function $y_0^i(x)$ no longer appears.

The two curves (7.2) and (7.3) of different origin and different spaces are shown to be the same unique curve. The curve C which in space C^m is the one and only one invariant of the contraction mapping is in the space R^{m+1} the limit of a sequence of curves generated by the solution of a system of integral equations.

III. METHODOLOGICAL DEVELOPMENT

The method evolved in this part in the solution of any system of differential equations which fulfill the conditions specified by the hypothesis in the extended existence theorem, will depend in part on the mode of choice of the initial arbitrary function $y_0^i(x)$. Two versions of the same method will be developed. The process is one of many possible modes of solution justified by the extended theorem.

The method to be discussed has been used [5] in the iterative solution of the N-body problem but with a specialized existence theorem formulated for the special case. This part of the paper will consist in the discussion of methodology followed by the formulation of a difference and an error function for the solution of the system of equations.

The Methodology: It is assumed that for the given system of differential equations and boundary conditions

(8.1) $\dfrac{dy^i}{dx} = f^i(x, y^1, ---, y^m)$; $[y^i(x) = y^i_{(0)}]$; $i = 1, ---, m$,

the hypothesis involved in the extended existence theorem is satisfied. The solution $y(x)$ is continuous on $|x - x_0| \leq h$ and is given as

$$y(x) = \lim_{n \to \infty} y_n^i(x)$$

where the continuous function $y_n^i(x)$ is an element of the complete metric space C^m. This element is generated by a contraction mapping of C^m into itself such that an iterative sequence $[y_0^i(x), y_1^i(x), ---]$ is formed, each element of which is dependent on the initial function $y_0^i(x)$. A variety of modes of definition may be used to formulate this initial function.

Consider the three distinct sets of solutions

(8.1)' $$y^i(x) = y^i_{(0)} + \int_{x_0}^{x} f^i(v, y^1, ---, y^m)\, dv, \quad i = 1, ---, m$$

(8.2)' $$y^i_\gamma(x) = y^i_{(0)} + \int_{x_0}^{x} f^i(v, \bar{y}^1_\gamma, ---, \bar{y}^m_\gamma)\, dv, \quad \gamma = n + \beta, \ \beta$$

(8.3)' $$y^i_n(x) = y^i_{(0)} + \int_{x_0}^{x} f^i(v, y^1_n, ---, y^m_n)\, dv, \quad n = 0, 1, 2, ---$$

respectively of the systems of differential equations

(8.1) $$\frac{dy^i}{dx} = f^i(x, y^1, ---, y^m)$$

(8.2) $$\frac{d\bar{y}^i_\gamma}{dx} = f^i(x, \bar{y}^1_\gamma, ---, \bar{y}^m_\gamma)$$

(8.3) $$\frac{dy^i_n}{dx} = f^i(x, y^1_n, ---, y^m_n)$$

The functions $y^i(x), \bar{y}^i_\gamma, y^i_n(x)$ will be called respectively the solution or limit solution, the finite serial solution and the iterative solution of the system (8.1).

Two variations in the method to be presented are open in the generation of the solutions and their relations. The first variation is to start with the solutions (8.1)' --- directly and the second by considering the derivatives (8.1),--- .

(a) Let

(8.4) $$y^i(x) = \sum_{k=0}^{\infty} a^i_k x^k, \quad i = 1, ---, m; \ |x| \leq h$$

be the power series representation of the solution $y^i(x)$ of the system (8.1) on the interval $|x| \leq h$. The gerality is not diminished if we write $x_0 = 0$. The finite serial solution of (8.1) or in the form (8.2) may then be written as

(8.5) $$\bar{y}^i_\gamma(x) = \sum_{k=0}^{\gamma} a^i_k x^k, \quad \gamma = n + \beta, \ \beta$$ fixed positive integer.

F or each n, an element (a partial sum) in t he converging sequence

(8.5)' $$(\bar{y}^i_\beta, \bar{y}^i_{\beta+1}, \bar{y}^i_{\beta+2}, ---)$$

is generated by the expression (8.5). The limit of the sequence of approximate solutions $\bar{y}^i_\gamma(x)$ of the system (8.2) is the solution of (8.1), namely

(8.6) $$y^i(x) = \lim_{n\to\infty} \bar{y}^i_\gamma(x).$$

From (8.6) it follows that

(8.7) $$\frac{dy^i(x)}{dx} = \lim_{n\to\infty} \frac{d}{dx}\bar{y}^i_\gamma(x)$$

If the elements $y^i_n(x)$ in the fundamental sequence

(8.8)' $$[\, y^i_0(x), y^i_1(x), ---, y^i_n(x), --- \,]$$

are derived from the contraction mapping (8.3)' then by the extended existence theorem

(8.8) $$y^i(x) = \lim_{n\to\infty} y^i_n(x)$$

From (8.8) we get

(8.9) $$\frac{dy^i(x)}{dx} = \lim_{n\to\infty} \frac{dy^i_n(x)}{dx}$$

Equate the right member of (8.6) and (8.8), namely

(8.10) $$\lim_{n\to\infty} y^i_n(x) = \lim_{n\to\infty} y^i_{n+\beta}(x)$$

or in finite form

(8.10)' $$y^i_n(x) = y^i_{n+\beta}(x) + \epsilon^i_{n+\beta}(x), \text{ for any } i \text{ and } \beta \geq 0$$

For n=o define

(8.11) $$[\,\bar{y}^i_{n+\beta}(x)]_{n=0} \equiv \bar{y}^i_\beta(x) = y^i_0(x) \;,\; [\,\epsilon^i_{n+\beta}(x)]_{n=0} \equiv \epsilon^i_\beta(x) = 0$$

for all x in its interval. It further follows from (8.10) that

(8.11)' $$\lim_{n\to\infty} \epsilon^i_{n+\beta}(x) = 0 \;, \text{ for any } i \text{ and } \beta \geq 0$$

Equation (8.10)' allows a determination of the iterative solution $y^i_n(x)$ of the seqıence (8.8)' provided the initial function $y^i_0(x)$ is given. A specialized form of this function is given by (8.11), namely $y^i_0(x) = \bar{y}^i_\beta(x)$. It is manifest that the larger the value of β, the closer is the iterative s olution to the actual solution $y^i(x)$ of the original system (8.1), for a vixed n. It is also obvious that the mode of generation of the initial function $\bar{y}^i_\beta(x) = y^i_0(x)$ by a finite power series is but one of many possible ways in its generation.

(b) The second mode in the formulation of the s olution $y^i(x)$

of the system (8.1) is to start with this given differential system and to equate the limits as given by (8.7) and (8.9). Thus

$$\lim_{n\to\infty} \frac{dy^i(x)}{dx} = \lim_{n\to\infty} \frac{d\bar{y}^i_\gamma(x)}{dx} \tag{8.12}$$

or in the equivalent form

$$\lim_{n\to\infty} f^i(x, y^1_n, ---, y^m_n) = \lim_{n\to\infty} f^i(x, \bar{y}^1_\gamma, ---, \bar{y}^m_\gamma), \quad i=1,---,m. \tag{8.13}$$

In finite terms this becomes

$$f^i(x, y^1_n, ---, y^m_n) = f^i(x, y^1_\gamma, ---, y^m_\gamma) + [\epsilon^i_\gamma(x)]', \quad i=1,---,m \tag{8.14}$$

where

$$\lim_{n\to\infty} [\epsilon^i_\gamma(x)]' = \frac{d}{dx} \lim_{n\to\infty} \epsilon^i_\gamma(x) = 0 \tag{8.15}$$

The expression (8.14) is a system of m equations which may be solved for the m unknown functions $y^1_n(x), ---, y^m_n(x)$ in terms of the remaining quantities $x, \bar{y}^1_\gamma, ---, \bar{y}^m_\gamma, (\epsilon^1_\gamma)', ---, (\epsilon^m_\gamma)'$

Thus the approximate solution is given by

$$y^i_n(x) = \Phi[x, \bar{y}^1_\gamma, ---, \bar{y}^m_\gamma, (\epsilon^1_\gamma)', ---, (\epsilon^m_\gamma)'] = \mathcal{Y}^i_{n+\beta}(x); \; i=1,---,m \tag{8.16}$$

where the order of the approximation is given by the value assigned to β which determines the initial function $y^i_0(x) = \bar{y}^i_\beta(x)$. This second process has been used in ref. [5] to generate the iterative solution of the analytic N-body problem.

For any specific problem the inverse form of (8.16) may be found more amenable to resolution, namely the expression of in terms of $\bar{y}^1_\gamma, ---, \bar{y}^m_\gamma$ in terms of $x, y^1_n, ---, y^m_n, (\epsilon^1_\gamma)', ---, (\epsilon^m_\gamma)'$. Thus

$$\bar{y}^i_\gamma(x) = \Psi[x, y^1_n, ---, y^m_n, (\epsilon^1_\gamma)', ---, (\epsilon^m_\gamma)'] \; ; \; i=1,---,m \tag{8.17}$$

where the quantities $[\epsilon^i_\gamma(x)]'$ still have the same meaning as that given by (8.15).

The Difference and Error Functions: In the previous section the function $\epsilon^{i}_{n+\beta}(x)$ was defined by the expression (8.10)'. Our purpose is to characterize this function and to get its measure. This will allow us to determine the rate of convergence of the iterative approximations $y^{i}_{n}(x)$ to the actual solution $y^{i}(x)$ of the system of differential equations (8.1).

Suppose the quantity $\epsilon^{i}_{n+\beta}(x)$ is given a more consistent symbolism relative to its definition (8.10); namely

(9.1) $$y^{i}_{n}(x) = \bar{y}^{i}_{n+\beta}(x) + \epsilon^{i}_{n,n+\beta}(x), \quad |x| \leq h$$

The quantity $\epsilon^{i}_{n,n+\beta}$ will be called the ϵ or the difference function between the two elements y^{i}_{n} and $\bar{y}^{i}_{n+\beta}$ of the respective contraction $(y^{i}_{0}, y^{i}_{1}, ---, y^{i}_{n}, ---)$ and partial sums $(\bar{y}^{i}_{\beta}, \bar{y}^{i}_{\beta+1}, ---, \bar{y}^{i}_{\beta+n}, ---)$ converging sequences.

For n=o define, as has been done, the ϵ-function by the expression

(9.2) $$\left[\epsilon^{i}_{n+\beta}(x)\right]_{n=0} \equiv \epsilon^{i}_{\beta}(x) = 0 \qquad \text{for any } i,\ |x| \leq h$$

It follows from (9.1) and definition (8.5) that

(9.3) $$\lim_{n\to\infty} \epsilon^{i}_{n+\beta}(x) = 0 \qquad \text{for any } i,\ |x| \leq h$$

An expression for the difference function is readily derived in terms of the metric of the complete metric space C^{m}. By means of definition (9.1), the statement

$$\max_{x,i}\left|y^{i}_{n}(x) - \bar{y}^{i}_{\gamma}(x)\right| = \max_{x,i}\left|\epsilon^{i}_{n,\gamma}(x)\right|, \quad \gamma = n+\beta$$

is valid. In view of the definition (5.4) of the metric ρ between two elements $y^{i}_{n}, \bar{y}^{i}_{\gamma} \in C^{m}$, it follows that

(9.4) $$\rho_{n,\gamma} = \max_{x,i}\left|\epsilon^{i}_{n,\gamma}(x)\right|$$

where $\rho_{n,\gamma}$ is defined as

(9.4)' $\rho_{n,\gamma} = \rho(y_n^i, \bar{y}_\gamma^i$

An inequality is thus generated from expression (9.1) with the use of (9.4), namely

(9.5) $y_n^i(x) \leq y_\gamma^i(x) + \rho_{n,\gamma}$

A useful formula (a generalization of the inequality (5.12)') in the comparison of any two elements y_n^i and y_{n+r}^i, $r = 0, 1, 2, ---$ of the iterative sequence $[y_0(x), y_1(x), ---]$ is unfolded from the definition

$$|y_n^i(x) - y_{n+r}^i(x)| = \epsilon_{n,n+r}^i(x), \quad r = 0, 1, 2, ---$$

It follows that

$$max_{x,i}|y_n^i - y_{n+r}^i| = max_{x,i}|\epsilon_{n,n+r}^i|$$

But by the definition of contraction mapping A of C^m into itself, namely

$$\rho(Ay_{n-1}^i, Ay_{n+r-1}^i) \leq \alpha\, \rho(y_{n-1}^i, y_{n+r-1}^i); \; y_{n-1}^i, y_{n+r-1}^i \in C^m; \; \alpha = Mh < 1$$

and since

$$\rho_{n,n+r} \equiv \rho(y_n^i, y_{n+r}^i) = \rho(Ay_{n-1}^i, Ay_{n+r-1}^i)$$

it follows that

(9.6)' $\rho_{n,n+r} \leq \alpha\, \rho(y_{n-1}^i, y_{n+r-1}^i) = \alpha\, \rho_{n-1,n+r-1}\,; \; \alpha = Mh < 1$

We thus have that

(9.6) $max_{x,i}|\epsilon_{n,n+r}^i(x)| \leq \alpha\, \rho_{n-1,n+r-1}\,; \; \alpha = Mh < 1\,; \; r = 0, 1, 2, ---$

Expression (9.6) in turn may be put in the form

(9.7) $max_{x,i}|\epsilon_{n,n+r}^i(x)| \leq max_{x,i}|\epsilon_{n-1,n+r-1}^i(x)|$

Expression (9.6) or (9.7) is a generalization of the inequality (5.12)'. Thus if (5.12) is written in the form

$$\rho(Ay_n^i, Ay_{n-1}^i) \leq \alpha\, \rho(y_n^i, y_{n-1}^i)$$

where for n-1 we write n (n-1 → n). The above form may then be expressed as

$$\max_{x,i}\left|\epsilon^i_{n,n+1}(x)\right| \le \alpha \max_{x,i}\left|\epsilon^i_{n-1,n}(x)\right|$$

then if r=1 is substituted in expression (9.7), the same inequality for the difference function is formed.

Consider now the elements (the partial sums) of the sequence $(\bar{y}^i_\beta, \bar{y}^i_{\beta+1}, \bar{y}^i_{\beta+2}, \text{---})$ as generated by the finite power series (8.5). For any two elements

$$\bar{y}^i_{\gamma+r} = \sum_{k=0}^{\gamma+r} a^i_k x^k, \quad \bar{y}^i_\gamma = \sum_{k=0}^{\gamma} a^i_k x^k; \quad \bar{y}^i_{\gamma+r}, \bar{y}^i_\gamma \in C^m; \; r = 0,1,2,\text{---},$$

the difference is given by

$$\bar{y}^i_{\gamma+r} - \bar{y}^i_\gamma = \sum_{k=\gamma+1}^{\gamma+r} a^i_k x^k, \text{ for any } x \text{ and } i.$$

This leads to the inequality

$$\max_{x,i}\left|\bar{y}^i_{\gamma+r} - \bar{y}^i_\gamma\right| \le \sum_{k=\gamma+1}^{\gamma+r} |a^i_k|\,|x^k|$$

Since $|x| \le h$ and in view of the definition of the ϵ-function, it follows that

$$\max_{x,i}\left|\epsilon^i_{\gamma+r,\gamma}\right| \le \sum_{k=\gamma+1}^{\gamma+r} |a^i_k|\, h^k. \tag{9.8}$$

Combine inequalities (9.7) and (9.8) and transpose to get

$$\max_{x,i}\left|\epsilon^i_{n,n+r}(x)\right| \le Mh \max_{x,i}\left|\epsilon^i_{n-1,n+r-1}(x)\right| - \max_{x,i}\left|\epsilon^i_{\gamma+r,\gamma}\right| + \sum_{k=\gamma+1}^{\gamma+r} |a^i_k|\, h^k; \; i=1,\text{---},m; \; \gamma = n+\beta, \; \beta \ge 0; \; n=0,1,2,\cdots; \; \alpha = Mh < 1; \; |x| \le h, \tag{9.9}$$

In terms of the metric of the space, inequality (9.9) becomes

$$\rho_{n,n+r} \le Mh\,\rho_{n-1,n-1+r} - \rho_{\gamma,\gamma+r} + \sum_{k=\gamma+1}^{\gamma+r} |a^i_k|\, h^k \tag{9.10}$$

Expression (9.9) is a measure of the maximum difference between any two elements $y^i_n(x)$ and $y^i_{n+r}(x)$ of the iterative sequence $(y^i_0, y^i_1, \text{---})$ in terms of (1) the maximum difference of two elements y^i_{n-1} and y_{n-1+r}

of the same sequence (the immediate predecessors of the former elements), (2) the maximum difference tween two elements $\bar{y}^i_{\gamma+r}$, y^i_γ of the partial sum sequence $(\bar{y}^i_\beta, \bar{y}^i_{\beta+1}, ---)$ and (3) the finite sum of the product of the coefficients and the radius of convergence h of the series

$$y^i(x) = \lim_{n\to\infty} \bar{y}^i_\gamma(x) = \sum_{k=0}^{\infty} a^i_k x^k$$

It may be noted, briefly, that expression (9.9) may also be interpreted as an error function of the actual solution $y^i(x)$ of the given system of differential equations relative to any term in the iterative sequence. Thus if we define

$$\epsilon^i_\mu(x) = y^i(x) - y^i_\mu(x), \quad \mu \geq 0, \tag{9.11}$$

then since

$$\epsilon^i_{n,n+r}(x) = y^i_{n+r}(x) - y^i_n(x) + y^i(x) - y^i(x) = \epsilon^i_n(x) - \epsilon^i_{n+r}(x),$$

it follows that

$$max_{x,i}\left|\epsilon^i_n(x) - \epsilon^i_{n+r}(x)\right| = max_{x,i}\left|\epsilon^i_{n,n+r}\right. \tag{9.12}$$

If now expression (9.9) is used we get the error function

$$\begin{aligned} & max_{x,i}\left|\epsilon^i_n(x) - \epsilon^i_{n+r}(x)\right| \\ & \quad \leq Mh\, max_{x,i}\left|\epsilon^i_{n-1,n-1+r}\right| - max_{x,i}\left|\epsilon^i_{\gamma,\gamma+r}\right| + \sum_{k=\gamma+1}^{\gamma+r} |a^i_k|\, h^k \end{aligned} \tag{9.13}$$

That expression (9.11) is a measure of the error function relative to the actual solution $y^i(x$ is manifest, since the

$$\lim_{n\to\infty}\left|\epsilon^i_n(x) - \epsilon^i_{n+r}(x)\right| = \left|\lim_{n\to\infty} \epsilon^i_n(x) - \lim \epsilon^i_{n+r}(x)\right|$$

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